Geometry on the lines of polar spine spaces

Krzysztof Petelczyc
kryzpet@math.uwb.edu.pl

Krzysztof Prażmowski
krzypraz@math.uwb.edu.pl

Mariusz Żynel*
mariusz@math.uwb.edu.pl

University of Białystok
Institute of Mathematics

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Outline

- Some definitions
- Our motivations and the goal
- How it all is supposed to be achieved

Some definitions

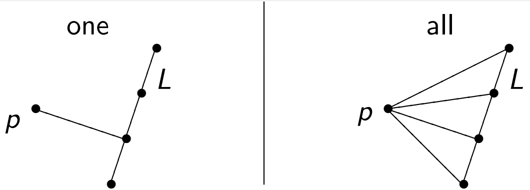
Polar spaces

Partial linear space

A partial linear space is a point-line structure $\langle S, \mathcal{L} \rangle$, where the elements of S are called points, the elements of $\mathcal{L}$ are called lines, and where $\mathcal{L} \subset 2^S$, if two distinct lines share at most one point and every line is of size (cardinality) at least 2.

One-or-all axiom

A point p, not on a line L, is collinear with one or all points of L.



Polar space

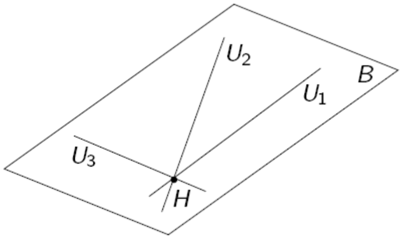
A polar space is a partial linear space satisfying one-or-all axiom.

Pencils

- V – a vector space of dimension n with $3 \leq n < \infty$
- $\mathrm{Sub}_k(V)$ – the set of all k-dimensional subspaces of V

For $H \in \mathrm{Sub}_{k-1}(V)$, $B \in \mathrm{Sub}_{k+1}(V)$ with $H \subset B$ a pencil of k-subspaces is a set of the form

$$\mathbf{p}(H, B) := \{U \in \mathrm{Sub}_k(V) \colon H \subset U \subset B\}.$$



Polar Grassmann spaces

- ξ – nondegenerate reflexive sesquilinear form of index r on V
- $\mathrm{Q}_k := \{ U \in \mathrm{Sub}_k(V) \colon \xi(U, U) = 0 \}$

The incidence structure $\langle \mathrm{Q}_1, \mathrm{Q}_2 \rangle$ is a polar space.

A point-line structure

$$\mathbf{P}_k(\mathrm{Q}) := \langle \mathrm{Q}_k,\ \mathcal{P}_k(\mathrm{Q}) \rangle,$$

where $\mathcal{P}_k(\mathrm{Q})$ is the family of all pencils of totally isotropic k-subspaces, is a polar Grassmann space.

Strong subspaces in the polar Grasmann space $\mathbf{P}_k(\mathrm{Q})$

- Every strong subspace is a projective space $\mathbf{P}$.
- There are two disjoint classes of maximal strong subspaces:

class	representative subspace	$\dim(\mathbf{P})$
stars	$[H, Y]_k \colon H \in \mathrm{Q}_{k-1}, Y \in \mathrm{Q}_r, H \subset Y$	$r - k$
tops	$[\Theta, B]_k \colon B \in \mathrm{Q}_{k+1}$	k

- Every line can be uniquely extended to a top but it is contained in more than one star.

New pointset, lineset and natural parallelizm

- W – a fixed subspace of V, $r_W = \operatorname{ind}(\xi\big|W)$
- m – a fixed integer such that $m \leq r_W \leq r - k + m$

New pointset:

$$\mathcal{F}_{k,m}(\mathrm{Q}, W) := \big\{U \in \mathrm{Q}_k\colon \dim(U \cap W) = m\big\}$$

- If $p \in \mathcal{P}_k(\mathrm{Q})$, then

$$\big|p \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)\big| \in \{0,1\} \quad \text{or} \quad |p \setminus \mathcal{F}_{k,m}(\mathrm{Q}, W)| \in \{0,1\}.$$

New lineset:

$$\mathcal{G}_{k,m}(\mathrm{Q}, W) := \big\{p \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)\colon p \in \mathcal{P}_k(\mathrm{Q}), |p \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)| \geq 2\big\}$$

Natural parallelizm for $L_1, L_2 \in \mathcal{G}_{k,m}(\mathrm{Q}, W)$:

$$L_1 \parallel L_2 :\iff L_1^\infty = L_2^\infty$$

Polar spine spaces

A point-line structure

$$\mathfrak{M} := \mathbf{A}_{k,m}(\mathrm{Q}, W) := \langle \mathcal{F}_{k,m}(\mathrm{Q}, W), \mathcal{G}_{k,m}(\mathrm{Q}, W), \parallel \rangle.$$

is a polar spine space.

- Lines of a polar spine space fall into three disjoint classes:

class	representative line $L = \mathbf{p}(H, B) \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)$	L^∞
affine	$H \in \mathcal{F}_{k-1,m}(\mathrm{Q}, W)$, $B \in \mathcal{F}_{k+1,m+1}(\mathrm{Q}, W)$	$H + (B \cap W)$
α-projective	$H \in \mathcal{F}_{k-1,m}(\mathrm{Q}, W)$, $B \in \mathcal{F}_{k+1,m}(\mathrm{Q}, W)$	–
ω-projective	$H \in \mathcal{F}_{k-1,m-1}(\mathrm{Q}, W)$, $B \in \mathcal{F}_{k+1,m+1}(\mathrm{Q}, W)$	–

- $\mathcal{L} := \mathcal{G}_{k,m}(\mathrm{Q}, W)$

Strong subspaces in the polar spine space $\mathbf{A}_{k,m}(\mathrm{Q}, W)$

- Every strong subspace of a polar spine space is a slit space, that is a projective space $\mathbf{P}$ with a subspace $\mathcal{D}$ removed.
- There are four disjoint classes of maximal strong subspaces:

class	representative subspace	
	$\dim(\mathbf{P})$	$\dim(\mathcal{D})$
ω-star	$[H, (H+W) \cap Y]_k \colon H \in \mathcal{F}_{k-1,m-1}(\mathrm{Q}, W), Y \in \mathrm{Q}_r, H \subset Y$	
	$\dim(W \cap Y) - m$	-1
α-star	$[H, Y]_k \cap \mathcal{F}_{k,m}(\mathrm{Q}, W) \colon H \in \mathcal{F}_{k-1,m}(\mathrm{Q}, W), Y \in \mathrm{Q}_r, H \subset Y$	
	$r - k$	$\dim(W \cap Y) - m - 1$
α-top	$[B \cap W, B]_k \colon B \in \mathcal{F}_{k+1,m}(\mathrm{Q}, W)$	
	$k - m$	-1
ω-top	$[\Theta, B]_k \cap \mathcal{F}_{k,m}(\mathrm{Q}, W) \colon B \in \mathcal{F}_{k+1,m+1}(\mathrm{Q}, W)$	
	k	$k - m - 1$

Intersections of stars and tops in a polar spine space

Lemma

A star and a top are disjoint, share a point or a line except an ω-star and an α-top that are disjoint or share a point. Two distinct tops are disjoint or share a point. Two distinct stars are disjoint or share a point, a line, a plane and so on up to a hyperplane.

	ω-star	α-star	α-top	ω-top
ω-star	any	$\emptyset$, point	$\emptyset$, point	$m = 0 \to \emptyset$, point $m > 0 \to \emptyset$, point, line
α-star		any	$\emptyset$, point, line	$\emptyset$, point, line
α-top			$\emptyset$, point, equal	$\emptyset$, point
ω-top				$\emptyset$, point, equal

Coplanarity and copencility

- A plane in $\mathfrak{M}$ is a 2-dimensional strong subspace in $\mathfrak{M}$.

Coplanarity

$$L_1 \; \pi \; L_2 \quad \text{iff} \quad \text{there is a plane } E \text{ such that } L_1, L_2 \subset E$$

- For a plane E in $\mathfrak{M}$ and $U \in \bar{E}$, the set

$$\mathbf{p}(U, E) := \left\{ L \in \mathcal{L} \colon U \in \bar{L} \subset \bar{E} \right\}$$

is a proper pencil if U is proper, or a parallel pencil otherwise.

Copencility

$$L_1 \; \rho \; L_2 \quad \text{iff} \quad \text{there is a proper pencil } p \text{ such that } L_1, L_2 \in p$$

Our motivations and the goal

The goal

Theorem

Assume that $\mathbf{A}_{k+1,m}(\mathrm{Q}, W)$ *is connected and the following two conditions are satisfied:*

$$4 \le k - m$$

and

$$r_W \le m + 3 \qquad \text{or} \qquad r_W \le k - 2m - 1.$$

Then the four structures:

- *the polar spine space* $\mathfrak{M} = \mathbf{A}_{k,m}(\mathrm{Q}, W)$,
- *the structure* $\mathbf{P}(\mathfrak{M})$ *of lines of* $\mathfrak{M}$ *and pencils of lines of* $\mathfrak{M}$,
- *the structure* $\langle \mathcal{L}, \boldsymbol{\pi} \rangle$ *of lines of* $\mathfrak{M}$ *and coplanarity relation,*
- *the structure* $\langle \mathcal{L}, \boldsymbol{\rho} \rangle$ *of lines of* $\mathfrak{M}$ *and copencility relation,*

are pairwise definitionally equivalent.

How it all is supposed to be achieved

Indispensable π-cliques and ρ-cliques

Flat A set
$$\mathsf{L}(E) := \{L \in \mathcal{L} \colon L \subset E\}$$
for some plane E in $\mathfrak{M}$.

Semiflat The set of all projective lines on a plane E in $\mathfrak{M}$ augmented with a maximal set of affine lines on E such that no two are parallel.

Semibundle A set
$$\mathsf{L}_U(X) := \{L \in \mathcal{L} \colon U \in \bar{L} \text{ and } L \subseteq X\}$$
for some strong subspace X of $\mathfrak{M}$ and $U \in \bar{X}$.

Tripod Three lines that meet in a point, are pairwise coplanar, and are not contained in a plane.

Maximal π-cliques and ρ-cliques

Proposition

A set K is a maximal π-clique iff either

- *K is a flat or*
- *K is a semibundle determined by at least 3-dimensional star or top*

Proposition

A set K is a maximal ρ-clique iff either

- *K is a semiflat or*
- *K is a proper semibundle determined by at least 3-dimensional star or top*

Definability of various π-cliques and ρ-cliques

$\mathbf{\Delta}^{s}_{\delta}(L_1, L_2, \ldots, L_s)$ iff

$\neq (L_1, L_2, \ldots, L_s)$, $L_i \;\delta\; L_j$ for all $i, j = 1, \ldots, s$ and

for all $M_1, M_2 \in \mathcal{L}$ if $M_1, M_2 \;\delta\; L_1, L_2, \ldots, L_s$ then $M_1 \;\delta\; M_2$

- $\mathbf{\Delta}^{3}_{\pi}(L_1, L_2, L_3)$ iff $\overline{L_1}, \overline{L_2}, \overline{L_3}$ form a tripod or a triangle and are not contained in two distinct maximal π-cliques
- $\mathbf{\Delta}^{3}_{\rho}(L_1, L_2, L_3)$ iff L_1, L_2, L_3 form a tripod or a triangle, are not on an affine plane, in case they are on a punctured plane one of L_1, L_2, L_3 is an affine line, and are not contained in two distinct maximal π-cliques

$$[L_1, L_2, \ldots, L_s]_{\delta} := \left\{ L \in \mathcal{L} \colon L \;\delta\; L_1, L_2, \ldots, L_s \right\}$$

$$\mathcal{K}^{s}_{\delta} := \left\{ [L_1, L_2, \ldots, L_s]_{\delta} \colon L_1, L_2, \ldots, L_s \in \mathcal{L} \text{ and } \mathbf{\Delta}^{s}_{\delta}(L_1, L_2, \ldots, L_s) \right\}$$

π-cliques spanned by three lines

Lemma

For each maximal π-clique K there is an integer s and there are lines $L_1, \ldots, L_s \in K$ such that

$$3 \le s \le r - k + 1, \quad \mathbf{\Delta}^s_{\pi}(L_1, \ldots, L_s), \quad \textit{and} \quad K = [\![L_1, \ldots, L_s]\!]_{\pi}.$$

The class of all maximal π-cliques is definable in $\mathfrak{M}$:

$$\mathcal{K}_{\pi} := \bigcup \left\{ \mathcal{K}^s_{\pi} \colon 3 \le s \le r - k + 1 \right\}.$$

Lemma

The family $\mathcal{K}^3_{\pi}$ consists of:

(i) *flats,*

(ii) *top semibundles,*

(iii) *semibundles contained in 3-dimensional stars,*

(iv) *semibundles contained in special ω-stars which do not intersect any other star in a 3-dimensional subspace.*

ρ-cliques spanned by three lines

Lemma

The family $\mathcal{K}^3_{\boldsymbol{\rho}}$ consists of:

(i) *projective flats,*

(ii) *punctured semiflats,*

(iii) *proper top semibundles,*

(iv) *proper semibundles contained in 3-dimensional stars,*

(v) *proper semibundles contained in special ω-stars.*

Pencils of lines and parallel pencils in $\mathfrak{M}$

$$\mathbf{p}_{\boldsymbol{\pi}}(L_1, L_2, L_3) \quad \text{iff} \quad L_1, L_2, L_3 \in K \text{ for some } K \in \mathcal{K}^3_{\boldsymbol{\pi}} \text{ and } \neg\boldsymbol{\Delta}_{\boldsymbol{\pi}}(L_1, L_2, L_3)$$

- $\mathbf{p}_{\boldsymbol{\pi}}(L_1, L_2, L_3)$ iff L_1, L_2, L_3 form a pencil of lines or a parallel pencil that is contained in a top or in a 3-dimensional star or in a special ω-star

$$\mathbf{p}_{\boldsymbol{\rho}}(L_1, L_2, L_3) \quad \text{iff} \quad L_1, L_2, L_3 \in K_1 \cap K_2 \text{ for some flat } K_1 \text{ and } K_2 \in \mathcal{K}^3_{\boldsymbol{\rho}} \text{ with } K_2 \nsubseteq K_1$$

- $\mathbf{p}_{\boldsymbol{\rho}}(L_1, L_2, L_3)$ iff L_1, L_2, L_3 form a pencil of lines

Proper semibundles and proper top semibundles

The family

$$\mathcal{B}_0 := \{K \in \mathcal{K}_\delta \colon \text{there is a pencil of lines } p \text{ such that} \\ p \subset K \text{ and } \dim(K) \geq 3\}$$

defined in $\langle \mathcal{L}, \boldsymbol{\delta} \rangle$ coincides with the family of all proper top semibundles, the family of all proper semibundles contained in special ω-stars, or the union of these two families depending on whether tops, stars or all of them as projective or semiaffine spaces are at least 4-dimensional.

Proper top semibundles $\mathcal{B}$ are distinguishable in $\mathcal{B}_0$ when:

$$4 \leq k - m$$

and

$$r_W \leq m + 3 \qquad \text{or} \qquad r_W \leq k - 2m - 1.$$

Grouping proper top semibundles into bundles

- For $K_i = \mathrm{L}_{U_i}(X_i) \in \mathcal{B}$, $i = 1, 2$ we define:

$$\Upsilon(K_1, K_2) \quad \text{iff} \quad (\exists\ L_1, L_2 \in K_1)(\exists\ M_1, M_2 \in K_2) \\ \big[L_1 \neq L_2 \wedge L_1\ \boldsymbol{\delta}\ M_1 \wedge L_2\ \boldsymbol{\delta}\ M_2\big].$$

Lemma

Let X_i be a top and $K_i = \mathrm{L}_{U_i}(X_i)$, $i = 1, 2$.
If $\Upsilon(K_1, K_2)$ and $K_1 \cap K_2 = \emptyset$, then $U_1 = U_2$.

Grouping proper top semibundles into bundles

- All tops are at least 4-dimensional spaces and thus no top is a punctured polar space.

Lemma

Let $X_i = [\Theta, B_i]_k$ be a top and $K_i = \mathrm{L}_{U_i}(X_i)$, $i = 1, 2$.
If $B_1 \perp B_2$ and $U_1 = U_2$, then $\Upsilon(K_1, K_2)$ and $K_1 \cap K_2 = \emptyset$.

- Assume that the polar spine space $\mathbf{A}_{k+1,m}(\mathrm{Q}, W)$ is connected.
- Let $\overline{\Upsilon}$ be the transitive closure of Υ.

Lemma

Let X_i be a top and $K_i = \mathrm{L}_{U_i}(X_i)$, $i = 1, 2$.
If $U_1 = U_2$, then $\overline{\Upsilon}(K_1, K_2)$ and $K_1 \cap K_2 = \emptyset$.

Bundles of lines

$$\Upsilon_{\emptyset}(K_1, K_2) \quad \text{iff} \quad \Upsilon(K_1, K_2), \Upsilon(K_2, K_1), \text{ and}$$
$$\text{either } K_1 \cap K_2 = \emptyset \text{ or } K_1 = K_2$$

Lemma

$\Upsilon_{\emptyset}(K_1, K_2)$ *iff* $U_1 = U_2$.

For $K \in \mathcal{B}$ we write

$$\Lambda_{\overline{\Upsilon_{\emptyset}}}(K) := \bigcup \{K' \in \mathcal{B} \colon \overline{\Upsilon_{\emptyset}}(K, K')\}.$$

Lemma

If U is a point and X is a top such that $U \in X$, then
$$\Lambda_{\overline{\Upsilon_{\emptyset}}}(\mathrm{L}_U(X)) = \{L \in \mathcal{L} \colon U \in L\}.$$

Thank you for your attention